Algebraic Geometry
B.Math. (Hons.) IIIrd year
Midsemestral exam
Second semester 2016
Instructor : B.Sury
Maximum marks 60

Q 1.
(4 marks) (a) Let I be an ideal in a commutative ring A. Prove I is primary if the radical is a maximal ideal.
(4 marks) (b) Give an example of I in a ring such that the radical of I is prime but I is not primary.
(4 marks) (c) Let A be a Noetherian ring and $P \subset Q$, $P \neq Q$ be prime ideals of A. Consider the A-module $M := A \times A$ and the A-submodule $N := P \times Q$. Show that N is not a primary submodule of M.
Hint: Consider the homothesy by an element of Q outside P.
(3 marks) (d) Let A be a Noetherian ring and M, a finitely generated module. Let $x \in M$ be so that $ann(x)$ is an ideal maximal with respect to inclusion among ideals of the form $ann(y)$ with $y \in M$. Prove that $ann(x) \in Ass(M)$.
(3 marks) (e) Let A be a Noetherian ring and let N be a primary submodule of a module M. If P is the set of $a \in A$ such that the homothesy

$$a_{M/N} : M/N \to M/N; \; x \mapsto ax$$

is not 1-1, prove that P is a prime ideal.
(2 marks) (f) In the problem (e), if $M = A$, prove that $P = \sqrt{N}$.

Q 2.
(4 marks) (a) Let K be a field. Write down the closed sets in the affine line $\mathbf{A}^1_K$. Deduce that the affine line is not Hausdorff with respect to the Zariski topology.
(4 marks) (b) Let K be a field and $W \subset \mathbf{A}^n_K$. Consider the ideal $I(W)$ in $K[X_1, \cdots, X_n]$. Show that

$$V(I(W)) := \{x \in K^n : f(x) = 0 \; \forall \; f \in I(W)\}$$

is the closure of W with respect to the Zariski topology.
(4 marks) (c) Let I be a prime ideal in $K[X_1, \cdots, X_n]$. Prove that $W =$

$V(I) \subset \mathbf{A}^n_K$ is irreducible.
(4 marks) (d) If A is a commutative ring, prove that a point in $Spec(A)$ is closed in the Zariski topology if and only if it corresponds to a maximal ideal.
(4 marks) (e) Let A be a Boolean ring (a ring in which $a^2 = a$ for each $a \in A$). If $a \in A$, then prove that the open set

$$D_a := \{P \in Spec(A) : a \notin P\}$$

is also closed.

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Q 3.
(5 marks) (a) Let $A \subseteq B$ be an integral domains where B is integral over A. If B is a field, prove that A must also be a field.
(7 marks) (b) If $\theta : A \to B$ is a homomorphism of finitely generated K-algebras, and $\mathbf{m}$ is a maximal ideal of B, prove that $\theta^{-1}(\mathbf{m})$ is a maximal ideal of A.
(8 marks) (c) Let K be an algebraically closed field. Let I be a proper ideal of $K[X_1, \cdots, X_n]$. Use Noether normalization for the K-algebra $K[X_1, \cdots, X_n]/I$ to deduce that the maximal ideals of this K-algebra correspond to points of $V(I)$.